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Generalized Coulomb Barrier Transmission Coefficient for Solar Neutrino and Astrophysical Problems

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For astrophysical calculations, it is customary to extrapolate higher-energy ($\gtrsim 20$ keV) data using the Gamow transmission coefficient in estimating the nonresonance nuclear fusion reaction cross sections $\sigma(E)$ for charged particles at low energies (<20 keV). We present a general extrapolation method based on a more realistic Coulomb barrier transmission coefficient, which can accommodate simultaneously both nonresonance and resonance contributions.

The solar neutrino flux is calculated using low-energy nuclear fusion cross sections $\sigma(E)$ as input data. Since $\sigma(E)$ values at solar energies ($\lesssim 20$ keV) cannot be measured in the laboratory, they are extracted from the laboratory measurements of $\sigma(E)$ at higher energies by an extrapolation procedure based on nuclear theory. However, the energy dependence of the nuclear reaction cross section $\sigma(E)$ cannot be obtained rigorously from first principles, since the many-nucleon scattering problem cannot be solved exactly even if the nucleon–nucleon force is given. Therefore, one must rely on physically reasonable model-dependent parametrization procedure based on a barrier transmission model (BTM). Such a procedure has been used extensively in astrophysical problems (Fowler *et al.*, 1967) involving the Gamow transmission coefficient for the Coulomb barrier (Gamow, 1928; Blatt and Weisskopf, 1952). In this paper, we present a more general and realistic barrier transmission model which can accommodate simultaneously both nonresonance and resonance contributions for extrapolating $\sigma(E)$ to lower energies.

The experimental results from 1968 to 1986 from the ^{37}Cl neutrino detector (the world's only solar neutrino detector in that period) in the

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Homestake Mine (Davis, 1986, 1987) initiated one of the most puzzling and long-lasting problems of modern physics, known as the missing solar neutrino flux problem, or more simply the solar neutrino problem. The processes $p + {}^7\text{Be} \rightarrow {}^8\text{B} + \gamma$ and ${}^8\text{B} \rightarrow {}^8\text{Be}^* + e^+ + \nu_e(<15\,\text{MeV})$ produce neutrinos to which the ${}^{37}\text{Cl}$ detector (Rowley, 1985; Davis, 1986, 1987) at Homestake Mine is sensitive. The average total rates of solar neutrino (electron type, ν_e) interactions $R^{\text{Cl}}_{\nu_e}(\text{exp})$ have been measured there by means of the reaction $\nu_e + {}^{37}\text{Cl} \rightarrow e^- + {}^{37}\text{Ar}$ from 1970 to the present. More recently, a real-time, directional solar-neutrino signal has been observed in the water Cherenkov detector Kamiokande-II (KAM-II) (Hirata *et al.*, 1989), which is sensitive mostly to ${}^8\text{B} \rightarrow {}^8\text{Be}^* + e^+ + \nu_e(<15\,\text{MeV})$. Of the many experiments that have been conducted, the experimental neutrino deficit is a factor of 2–3 times lower than the accepted prediction from the standard solar model (SSM) (Bahcall and Ulrich, 1988; Bahcall *et al.*, 1988; Bahcall, 1989). The much newer ${}^{71}\text{Ga}$ detectors in the Gran Sasso Laboratory in Italy (GALLEX collaboration) (Anselmann *et al.*, 1992) and for the Soviet–American Gallium Experiment (SAGE) at Baksan in the former Soviet Union (Abazov *et al.*, 1991; Garvin *et al.*, 1992) have better detection efficiency. Although the new detectors are lessening the deficit, it has not disappeared.

The SSM has been successful in relating the mass and composition of the sun to its luminosity and lifetime. The SSM has also been widely accepted, as it appears to be based upon well-understood nuclear physics. However, this has included approximations that are inconclusively established both for higher energies and for the solar energy regime. In fact, the SSM has appeared to work so well that the preponderance of attempted theoretical solutions have been directed at the neutrinos, rather than nuclear physics input for the SSM. Of many proposed hypotheses for solving the solar neutrino problem, the neutrino oscillation hypothesis appears to be most popular (Kuo and Pantaleone, 1989; Bahcall, 1989). However, it is also desirable to reexamine the accuracies of the nuclear physics input.

Previous low-energy ($<20\,\text{keV}$) $\sigma(E)$ for nonresonance reactions involving charged particles used in the standard solar model calculations (Bahcall and Ulrich, 1988; Bahcall and Pinsonneault, 1992) are calculated by extrapolating the experimental values of $\sigma(E)$ at higher energies using the parametrization (Fowler *et al.*, 1967)

$$\sigma(E) = \frac{S(E)}{E} T_{\text{G}}(E) \tag{1}$$

where $T_{\text{G}}(E) = \exp[-(E_{\text{G}}/E)^{1/2}]$, $E_{\text{G}} = (2\pi\alpha Z_1 Z_2)^2 \mu c^2/2$, with the reduced mass $\mu = m_1 m_2/(m_1 + m_2)$ and E is the center-of-mass (CM) kinetic energy.

The transmission coefficient ("Gamow" factor) $T_G(E)$ results from the approximation $E \ll B$ (Coulomb barrier height).

In order to generalize the conventional Gamow transmission coefficient, we introduce for the fusing system the following potential, which consists of an interior square-well nuclear potential and an exterior repulsive potential:

$$V(r) = \begin{cases} -V_0 & r < R \\ Z_1 Z_2 e^2/r, & r \geq R \end{cases} \tag{2}$$

For the potential barrier given by equation (2), an approximate S-wave ($l = 0$) solution for $T(E)$ can be calculated in the Wentzel–Kramers–Brillouin (WKB) approximation as (Kim *et al.*, 1993)

$$\begin{aligned} T_R^{\text{WKB}}(E) &= \exp\left\{-2\left(\frac{2\mu}{\hbar^2}\right)^{1/2} \int_R^{r_a} \left(\frac{Z_1 Z_2 e^2}{r} - E\right)^{1/2} dr\right\} \\ &= \exp\left(-\left(\frac{E_G}{E}\right)^{1/2} \left(\frac{2}{\pi}\right)\left\{\cos^{-1}\left[\left(\frac{E}{B}\right)^{1/2}\right] - \left(\frac{E}{B}\right)^{1/2}\left(1 - \frac{E}{B}\right)^{1/2}\right\}\right) \end{aligned} \tag{3}$$

where $B = Z_1 Z_2 e^2/R$ and r_a is the classical turning point, $Z_1 Z_2 e^2/r_a = E$. Note that $T_R^{\text{WKB}}(E)$ is defined only for $E \leq Z_1 Z_2 e^2/R$ (Coulomb barrier height) and $T_R^{\text{WKB}}(Z_1 Z_2 e^2/R) = 1$. The traditional Gamow transmission coefficient used in equation (1) can be obtained from equation (3) with $R = 0$ (or equivalently $E \ll B$):

$$\begin{aligned} T_G(E) = T_R^{\text{WKB}}(E) &= \exp\left[-2\left(\frac{2\mu}{\hbar^2}\right)^{1/2} \int_0^{r_a} \left(\frac{Z_1 Z_2 e^2}{r} - E\right)^{1/2} dr\right] \\ &= \exp\left[-\left(\frac{E_G}{E}\right)^{1/2}\right] \end{aligned} \tag{4}$$

To obtain improved and more general transmission coefficients, we use partial wave solutions of the Schrödinger equation. For the potential described by equation (2), a general solution of the Schrödinger equation for the exterior wave function in the exterior region ($r \geq R$) is given by (Blatt and Weisskopf, 1952)

$$u_l^{\text{ext}}(r) = a u_l^{(-)}(r) + b u_l^{(+)}(r) \tag{5}$$

where

$$u_l^{(+)}(r) = [\exp(-i\delta_l^C)][G_l(r) + iF_l(r)] \tag{6}$$

δ_l^C is the Coulomb phase shift and $u_l^{(-)}$ is complex conjugate of $u_l^{(+)}$. Here F_l and G_l are the regular and irregular Coulomb wave functions normalized asymptotically ($r \to \infty$) as

$$F_l(r) \approx \sin[kr - l\pi/2 - \eta ln(2kr) + \delta_l^C]$$
$$G_l(r) \approx \cos[kr - l\pi/2 - \eta ln(2kr) + \delta_l^C] \tag{7}$$

where η is the Sommerfeld parameter, $\eta = Z_1 Z_2 e^2/\hbar v$, and k is related to E by $E = \hbar^2 k^2/2\mu$.

For the interior region ($R_1 \leq r < R$), where R_1 is the range of an effective short-range repulsive potential for $l > 0$ attributable to the repulsive nucleon–nucleon interaction at short distances less than ~ 0.3 fm, a general solution of the Schrödinger equation for the interior wave function is

$$u_l^{\text{int}}(r) = u_l^{(-)\text{int}}(r) + c_l u_l^{(+)\text{int}}(r) \tag{8}$$

with

$$u_l^{(-)\text{int}}(r) = -[\tilde{n}_l(Kr) + i\tilde{j}_l(Kr)] \tag{9}$$

and

$$u_l^{(+)\text{int}}(r) = -[\tilde{n}_l(Kr) - i\tilde{j}_l(Kr)] \tag{10}$$

with $\tilde{j}_l(x) = x j_l(x)$ and $\tilde{n}_l(x) = x n_l(x)$, where $j_l(x) = (\pi/2x)^{1/2} J_{l+1/2}(x)$ and $n_l(x) = (-1)^{l+1}(\pi/2x)^{1/2} J_{-l-1/2}(x)$ are the spherical Bessel function and spherical Newmann function, respectively. $J_{l+1/2}(x)$ is an ordinary Bessel function of half-odd-integer order and $\hbar^2 K^2/2\mu = V_0 + E$ with $E = \hbar^2 k^2/2\mu$. We introduce two real parameters τ_l and ϕ_l and write $c_l = \tau_l e^{i\phi_l}$, ($\tau_l \leq 1$).

Using the boundary condition at $r = R$ [i.e., matching the logarithmic derivatives of equations (5) and (8)], we obtain the barrier transmission coefficient $T_l(E) = 1 - |b_l/a_l|^2$:

$$T_l(E) = \frac{-4 s_l \mathscr{I} P_l}{|(\Delta_l + i s_l) - (\mathscr{R} P_l + i \mathscr{I} P_l)|^2} \tag{11}$$

where $\mathscr{R}P_l$ and $\mathscr{I}P_l$ are the real and imaginary parts, respectively, of the logarithmic derivative P_l of $u_1^{\text{int}}(r)$:

$$\begin{aligned} P_l &= R \left.\frac{du_l^{\text{int}}/dr}{u_l^{\text{int}}}\right|_{r=R-} \\ &= R \frac{\{[d\tilde{n}_l(Kr)/dr](1 + c_l) + i[d\tilde{j}_l(Kr)/dr](1 - c_l)\}_{r=R-}}{\tilde{n}_l(KR)(1 + c_l) + i\tilde{j}_l(KR)(1 - c_l)} \\ &= \mathscr{R}P_l + i\mathscr{I}P_l \qquad (\mathscr{I}P_l < 0) \end{aligned} \tag{12}$$

The quantities s_l and Δ_l in equation (11) are defined as

$$s_l = R[(G_l F_l' - F_l G_l')/(G_l^2 + F_l^2)]_{r=R} \tag{13}$$

and

$$\Delta_l = R[(G_l G_l' + F_l F_l')/(G_l^2 + F_l^2)]_{r=R} \tag{14}$$

respectively.

Using the new lth partial wave transmission coefficient $T_l(E)$, equation (11), we can write a more general parametrization of $\sigma(E)$ in terms of the lth partial wave cross section $\sigma_l(E)$ for the fusing system described by the potential (2) as

$$\sigma(E) = \frac{\pi}{k^2} \sum_l (2l+1) S_l(E) T_l(E) = \sum_l \sigma_l(E) \tag{15}$$

where $S_l(E)$ is the lth partial wave S-factor and

$$\sigma_l(E) = \frac{\tilde{S}_l(E)}{E} T_l(E) \tag{16}$$

If the lowest partial wave ($l = 0$) contribution is expected to be dominant for low energies ($\lesssim 20$ keV), then the total cross section $\sigma(E)$ is given by

$$\sigma(E) \approx \sigma_0(E) = \frac{\tilde{S}_0(E)}{E} T_0(E) \tag{17}$$

and the transmission coefficient is given by $T_0(E) = 1 - |b_0/a_0|^2$:

$$T_0(E) = \frac{-4 s_0 \mathscr{I} P_0}{|(\Delta_0 + i s_0) - (\mathscr{R} P_0 + i \mathscr{I} P_0)|^2} \tag{18}$$

where $\mathscr{R}P_0$, $\mathscr{I}P_0$, s_0, and Δ_0 are given by equations (12)–(14) with $l = 0$. The explicit form of $T_0(E)$, equation (18), is

$$T_0(E) = \frac{4 s_0 \bar{K}_1 R}{|(\Delta_0 + i s_0) - (\bar{K}_2 R - i \bar{K}_1 R)|^2} \tag{19}$$

where

$$s_0 = R[(G_0 F_0' - F_0 G_0')/(G_0^2 + F_0^2)]_{r=R} \tag{20}$$

$$\Delta_0 = R[(G_0 G_0' + F_0 F_0')/(G_0^2 + F_0^2)]_{r=R} \tag{21}$$

$$\bar{K}_1 = \frac{K(1 - \tau_0^2)}{1 + 2\tau_0 \cos(2KR + \phi_0) + \tau_0^2} \tag{22}$$

and

$$\bar{K}_2 = \frac{-2K\tau_0 \sin(2KR + \phi_0)}{1 + 2\tau_0 \cos(2KR + \phi_0) + \tau_0^2} \tag{23}$$

$T_0(E)$, equation (19), described by four parameters, V_0, R, τ_0, and ϕ_0, contains both nonresonance and resonance contributions, and also the interference term between them. The four parameters can be determined from the cross section containing both a resonance part (resonance energy and width) and a nonresonance background.

$T_0(E)$ has a Breit–Wigner form when $(\Delta_0 + is_0) - (\bar{K}_2 R - i\bar{K}_1 R) = 0$ at a pole $E = E_R - i\Gamma/2$ in the complex E plane. The resonance energy E_R and width Γ are determined by the parameters τ_0 and ϕ_0 for fixed values of V_0 and R. The resonance behavior of $T_0(E)$ generated from fitting $\sigma(E)$ with particular values of parameters is a Coulomb barrier transmission (CBT) resonance due to an interplay of Coulomb barrier and nuclear interaction, and is to be distinguished from the conventional resonances such as narrow neutron capture resonances, which are primarily due to the nuclear interaction. The resonances present in $\sigma(E)$ which are shown by some related experiments to be of non-CBT type are to be treated by conventional methods. Very broad resonance behaviors for cross sections observed in many nuclear reactions (Chulick *et al.*, 1993), such as for reactions $^2H(D,p)^3H$, $^2H(D,n)^3He$, $^3He(D,p)^4He$, and $^3H(D,n)^4He$, may correspond to CBT resonances and may yield different low-energy extrapolations from those obtained by the use of the conventional transmission coefficient $T_G(E)$, since the low-energy tail of the CBT resonance is expected to be different from that of the conventional case.

For the case of nonresonance cross section, $\tau_0 = 0$, and $T_0(E)$, equation (19), reduces to the result given by Blatt and Weiskopf (1952),

$$T_{BW}(E) = \frac{4s_0 KR}{\Delta_0^2 + (s_0 + KR)^2} \tag{24}$$

It should be noted that $T_{BW}(E)$, equation (24), does not have a resonance structure, while $T_0(E)$, equation (19), does.

In the previous parametrizations of $\sigma(E)$, the resonance part of $\sigma(E)$ is parametrized with a Breit–Wigner resonance formula to be subtracted from the experimental data (Fowler *et al.*, 1967; Kim *et al.*, 1993) or included in $S(E)$ in equation (1) (Chulick *et al.*, 1993). The nonresonance formula (1) is then used to fit the resultant "data". Our more general formula for $T_0(E)$, equation (19), with equation (17), will allow us to parametrize the experimental data exhibiting the CBT resonance behavior by the same formula, equation (17), thus avoiding separate use of the Breit–Wigner formula for subtracting the resonance contribution from $\sigma(E)$. Furthermore, the interference term between the resonance and nonresonance contributions is automatically included in equations (17) and (19). The formulation described by equations (15)–(17) and (19) is generalization of equation (1) and thus can provide a more realistic and general parametrization method for low-energy nuclear fusion cross sections needed for solar neutrino and astrophysical calculations.

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